

ON PRINCIPAL BUNDLES OVER A PROJECTIVE VARIETY DEFINED OVER A FINITE FIELD

INDRANIL BISWAS

ABSTRACT. Let M be a geometrically irreducible smooth projective variety, defined over a finite field k , such that M admits a k -rational point x_0 . Let $\varpi(M, x_0)$ denote the corresponding fundamental group-scheme introduced by Nori. Let E_G be a principal G -bundle over M , where G is a reduced reductive linear algebraic group defined over the field k . Fix a polarization ξ on M . We prove that the following three statements are equivalent:

- (1) The principal G -bundle E_G over M is given by a homomorphism $\varpi(M, x_0) \longrightarrow G$.
- (2) There are integers $b > a \geq 1$, such that the principal G -bundle $(F_M^b)^* E_G$ is isomorphic to $(F_M^a)^* E_G$, where F_M is the absolute Frobenius morphism of M .
- (3) The principal G -bundle E_G is strongly semistable, $\text{degree}(c_2(\text{ad}(E_G))c_1(\xi)^{d-2}) = 0$, where $d = \dim M$, and $\text{degree}(c_1(E_G(\chi))c_1(\xi)^{d-1}) = 0$ for every character χ of G , where $E_G(\chi)$ is the line bundle over M associated to E_G for χ .

In [16], the equivalence between the first statement and the third statement was proved under the extra assumption that $\dim M = 1$ and G is semisimple.

1. INTRODUCTION

Let M be a geometrically irreducible smooth projective variety, defined over a finite field k , such that M admits a k -rational point. Fix a k -rational point x_0 of M . Also, fix a polarization ξ on M in order to define the degree of torsionfree coherent sheaves on M .

Let G be a reduced reductive linear algebraic group defined over the field k . We recall that a principal G -bundle E_G over M is defined to be *strongly semistable* if $(F_M^n)^* E_G$ is semistable for all $n \geq 1$, where

$$F_M : M \longrightarrow M$$

is the absolute Frobenius morphism (see [15, p. 129, Definition 1.1] and [15, pp. 131–132, Lemma 2.1] for the definition of a semistable principal G -bundle). It is known that a principal G -bundle E_G is strongly semistable if and only if for every representation

$$\eta : G \longrightarrow \text{GL}(V),$$

where V is a finite dimensional k -vector space, satisfying the condition that η takes the connected component of the center of G to the center of $\text{GL}(V)$, the vector bundle

$$E_G(V) := E \times^G V \longrightarrow M$$

2000 *Mathematics Subject Classification.* 14L15, 14F05.

Key words and phrases. Fundamental group-scheme, principal bundle, finite field.

associated to E_G for η is semistable (see [14, p. 288, Theorem 3.23]). We are using the notation $E_G \times^G V$ of [6, p. 114, Définition 1.3.1] in order to denote the quotient of $E_G \times V$ by the “twisted” diagonal action of G ; this notation will be used throughout.

The notion of fundamental group–scheme was introduced by Nori [12]. The fundamental group–scheme of M with base point x_0 will be denoted by $\varpi(M, x_0)$. We recall that $\varpi(M, x_0)$ is the affine group–scheme over k associated to the neutral Tannakian category defined by the essentially finite vector bundles on M ; the fiber functor of the neutral Tannakian category sends an essentially finite vector bundle to its fiber over x_0 (see Section 3).

There is a tautological principal $\varpi(M, x_0)$ –bundle $E_{\varpi(M, x_0)}$ over M ; its description is recalled in Section 3. Therefore, given any homomorphism

$$(1.1) \quad \rho : \varpi(M, x_0) \longrightarrow G$$

we have the principal G –bundle

$$(1.2) \quad E_\rho := E_{\varpi(M, x_0)} \times^{\varpi(M, x_0)} G \longrightarrow M$$

which is obtained by extending the structure group of the tautological principal $\varpi(M, x_0)$ –bundle $E_{\varpi(M, x_0)}$ using ρ .

For any zero–cycle $C \in \text{CH}^0(M)$, let $[C] \in \mathbb{Z}$ be the degree of C .

Our aim is to prove the following theorem.

Theorem 1.1. *Let E_G be a principal G –bundle over M . The following three statements are equivalent.*

- (1) *There are integers $b > a \geq 1$, such that the principal G –bundle $(F_M^b)^* E_G$ is isomorphic to $(F_M^a)^* E_G$.*
- (2) *There is a homomorphism ρ as in Eq. (1.1) such that E_G is isomorphic to the principal G –bundle E_ρ defined in Eq. (1.2).*
- (3) *The principal G –bundle E_G is strongly semistable and the following two conditions hold:*

- *for each character χ of G , the line bundle $E_G(\chi) = E_G \times^G k \longrightarrow M$ associated to E_G for χ has the property that*

$$[c_1(E_G(\chi))c_1(\xi)^{d-1}] \in \mathbb{Z}$$

vanishes, where $d = \dim M$, and

- *$[c_2(\text{ad}(E_G))c_1(\xi)^{d-2}] = 0$, where $\text{ad}(E_G)$ is the adjoint vector bundle of E_G .*

In [16], the equivalence between the second and the third statements in Theorem 1.1 was proved under the extra assumption that $\dim M = 1$ and G is semisimple.

We also prove the following:

Corollary 1.2. *A vector bundle E over M is essentially finite if and only if there are integers $b > a \geq 1$, such that the vector bundle $(F_M^b)^* E$ is isomorphic to $(F_M^a)^* E$.*

2. STRONGLY SEMISTABLE PRINCIPAL BUNDLES

Let k be a finite field. Let M be a geometrically irreducible smooth projective variety defined over k . Let d be the dimension of M .

Fix a very ample line bundle ξ over M . For any torsionfree coherent sheaf V on M , define

$$\text{degree}(V) := [c_1(V)c_1(\xi)^{d-1}] \in \mathbb{Z}.$$

A vector bundle E over M is called *finite* if there are two distinct polynomials

$$f, g \in \mathbb{Z}[X]$$

with nonnegative coefficients such that the vector bundle $f(E)$ is isomorphic to $g(E)$ (see [13, p. 80, Lemma 3.1]). A vector bundle V over M of degree zero is called *essentially finite* if there is a finite vector bundle E and a subbundle $W \subset E$ such that

- the vector bundle W is of degree zero, and
- V is isomorphic to a quotient of W .

(See [13, p. 82, Definition].)

Let G be a reduced reductive linear algebraic group defined over k . For a principal G -bundle E_G over M , the *adjoint vector bundle* $\text{ad}(E_G)$ is the one associated to E_G for the adjoint action of G on its own Lie algebra.

Lemma 2.1. *Let E_G be a strongly semistable principal G -bundle over M satisfying the following two conditions:*

- *for each character χ of G , the line bundle $E_G(\chi) = E_G \times^G k \rightarrow M$ associated to E_G for χ has the property that*

$$[c_1(E_G(\chi))c_1(\xi)^{d-1}] \in \mathbb{Z}$$

vanishes, and

- *$[c_2(\text{ad}(E_G))c_1(\xi)^{d-2}] \in \mathbb{Z}$ vanishes.*

Let V be a finite dimensional representation of G over k . Then the associated vector bundle $E_G(V) = E_G \times^G V$ over M is essentially finite.

Proof. Since E_G is strongly semistable, the associated vector bundle $E_G(V)$ is also strongly semistable [14, p. 288, Theorem 3.23]. Taking the character χ in the statement of the lemma to be the one associated to the representation $\bigwedge^{\text{top}} V$ of G we conclude that $[c_1(E_G(V))c_1(\xi)^{d-1}] = 0$.

We will show that $[c_2(E_G(V))c_1(\xi)^{d-2}] = 0$. For that, fix a filtration of G -modules

$$(2.1) \quad 0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n := V$$

such that each successive quotient V_i/V_{i-1} , $i \in [1, n]$, is irreducible.

The center of G will be denoted by Z , and the Lie algebra of G will be denoted by \mathfrak{g} . We note that \mathfrak{g} is a faithful G/Z -module.

Since V_i/V_{i-1} in Eq. (2.1) is an irreducible G -module, the action of the center Z on V_i/V_{i-1} is multiplication by a fixed character χ_i of G . Hence Z acts trivially on

$$\text{End}(V_i/V_{i-1}) = (V_i/V_{i-1}) \bigotimes (V_i/V_{i-1})^*.$$

Therefore, the action of G on $\text{End}(V_i/V_{i-1})$ gives an action of G/Z on $\text{End}(V_i/V_{i-1})$. We will consider $\text{End}(V_i/V_{i-1})$ as a G/Z -module. Since \mathfrak{g} is a faithful G/Z -module, each G/Z -module V_i/V_{i-1} , $i \in [1, n]$, is a subquotient of a G/Z -module of the form

$$(2.2) \quad \mathcal{V} := \bigoplus_{i=1}^m \mathfrak{g}^{\otimes a_i} \otimes (\mathfrak{g}^*)^{\otimes b_i},$$

where a_i, b_i are nonnegative integers and $m > 0$ (see [1, Proposition 4.4]). We recall that a subquotient of \mathcal{V} is a sub G/Z -module of a quotient G/Z -module of \mathcal{V} (which is same as being a quotient of a sub module of \mathcal{V}).

Using [14, p. 288, Theorem 3.23] it follows that the adjoint bundle $\text{ad}(E_G)$, which is associated to E_G for the G -module \mathfrak{g} , is strongly semistable. Also, $\bigwedge^{\text{top}} E_G(\mathcal{V})$ is a trivial line bundle because the G -module $\bigwedge^{\text{top}} \mathcal{V}$ is trivial. Hence the vector bundle $E_G(\mathcal{V})$ associated to E_G for the G -module \mathcal{V} in Eq. (2.2) has the following properties:

- $E_G(\mathcal{V})$ is strongly semistable,
- $c_1(E_G(\mathcal{V})) = 0$, and
- $[c_2(E_G(\mathcal{V}))c_1(\xi)^{d-2}] = 0$ (recall that $[c_2(\text{ad}(E_G))c_1(\xi)^{d-2}] = 0$ by assumption).

Since $\text{End}(V_i/V_{i-1})$ is a subquotient of \mathcal{V} , there is a quotient G/Z -module

$$Q = \mathcal{V}/\mathcal{K}$$

of \mathcal{V} such that the G/Z -module $\text{End}(V_i/V_{i-1})$ is isomorphic to a submodule of Q . Let $E_G(\mathcal{K})$, $E_G(\text{End}(V_i/V_{i-1}))$ and $E_G(Q/\text{End}(V_i/V_{i-1}))$ be the vector bundles associated to E_G for the G -modules \mathcal{K} , $\text{End}(V_i/V_{i-1})$ and $Q/\text{End}(V_i/V_{i-1})$ respectively. So, as elements of the Grothendieck K -group $K(M)$,

$$(2.3) \quad E_G(\mathcal{V}) = E_G(\mathcal{K}) + E_G(Q/\text{End}(V_i/V_{i-1})) + E_G(\text{End}(V_i/V_{i-1})) \in K(M).$$

Since E_G is strongly semistable, using [14, p. 288, Theorem 3.23] we know that the associated vector bundles $E_G(\mathcal{K})$, $E_G(Q/\text{End}(V_i/V_{i-1}))$ and $E_G(\text{End}(V_i/V_{i-1}))$ are strongly semistable. Also $\bigwedge^{\text{top}} E_G(\mathcal{K})$, $\bigwedge^{\text{top}} E_G(Q/\text{End}(V_i/V_{i-1}))$ and $\bigwedge^{\text{top}} E_G(\text{End}(V_i/V_{i-1}))$ are trivial line bundles because \mathcal{K} , $\text{End}(V_i/V_{i-1})$ and $Q/\text{End}(V_i/V_{i-1})$ are all given by G/Z -modules, and G/Z does not have any nontrivial character. From Bogomolov's inequality, [3, p. 500, Theorem] (see also [9, p. 252, Theorem 0.1]), we know that for a strongly semistable vector bundle $W \rightarrow M$ with $\bigwedge^{\text{top}} W$ trivial,

$$[c_2(W)c_1(\xi)^{d-2}] \geq 0.$$

Since $[c_2(E_G(\mathcal{V}))c_1(\xi)^{d-2}] = 0$, from Eq. (2.3) we now have

$$[c_2(E_G(\text{End}(V_i/V_{i-1})))c_1(\xi)^{d-2}] \geq 0.$$

Consequently, $[c_2(E_G(\mathcal{V}))c_1(\xi)^{d-2}] \in \mathbb{Z}$ vanishes.

The semistable vector bundle $E \longrightarrow M$ of fixed rank with

$$[c_1(E)c_1(\xi)^{d-1}] = 0 = [c_2(E)c_1(\xi)^{d-2}]$$

form a bounded family [9, p. 269, Theorem 4.2]. The field k being finite, it follows from the above boundedness theorem that the set of vector bundles $\{(F_M^n)^*E_G(V)\}_{n \geq 1}$ contains only finitely many isomorphism classes. Hence there are integers $b > a \geq 1$ such that

$$(2.4) \quad (F_M^b)^*E_G(V) = (F_M^a)^*E_G(V).$$

Since $(F_M^{b-a})^*(F_M^a)^*E_G(V) = (F_M^b)^*E_G(V)$, from Eq. (2.4) we have

$$(2.5) \quad (F_M^{b-a})^*(F_M^a)^*E_G(V) = (F_M^a)^*E_G(V).$$

Consequently, from a theorem of Lange–Stuhler and Deligne we conclude the following:

There is an étale Galois covering

$$\phi : Y \longrightarrow M$$

such that the pull back $\phi^*(F_M^a)^*E_G(V)$ is a trivial vector bundle; see [8, p. 75, Theorem 1.4] and [11, § 3.2].

Now from [2, pp. 552–553, Proposition 2.3] it follows that the vector bundle $E_G(V)$ is essentially finite. This completes the proof of the lemma. \square

3. FUNDAMENTAL GROUP–SCHEME AND PRINCIPAL BUNDLES

Henceforth, we will assume that the variety M admits a k –rational point. Fix a k –rational point x_0 of M .

The fundamental group–scheme of M with base point x_0 will be denoted by $\varpi(M, x_0)$ (see [13, p. 85, Definition 1 and Proposition 2]). There is a tautological principal $\varpi(M, x_0)$ –bundle over M whose construction is recalled below. (See also [13, p. 84, Definition] for this tautological principal $\varpi(M, x_0)$ –bundle.)

The fundamental group–scheme $\varpi(M, x_0)$ is defined by giving the corresponding neutral Tannakian category. More precisely, consider the neutral Tannakian category defined by the essentially finite vector bundles over M (their definition was recalled in Section 2); the fiber functor for the neutral Tannakian category sends an essentially finite vector bundle W to the k –vector space W_{x_0} , where W_{x_0} is the fiber of W over the base point x_0 . The fundamental group–scheme $\varpi(M, x_0)$ is defined to be the group–scheme associated to this neutral Tannakian category. Consequently, each essentially finite vector bundle W over M gives a finite dimensional representation of $\varpi(M, x_0)$ over k such that the underlying k –vector space is W_{x_0} .

Let $\text{Rep}(\varpi(M, x_0))$ denote the neutral Tannakian category defined by the representation of $\varpi(M, x_0)$. So $\text{Rep}(\varpi(M, x_0))$ is equivalent to the above neutral Tannakian category defined by the essentially finite vector bundles over M . Let $\text{Vect}(M)$ denote the category of vector bundles over X . We have a tautological functor

$$(3.1) \quad \mathcal{F} : \text{Rep}(\varpi(M, x_0)) \longrightarrow \text{Vect}(M)$$

that sends any $W \in \text{Rep}(\varpi(M, x_0))$ to the essentially finite vector bundle W . Using [5, p. 149, Theorem 3.2], [13, Lemma 2.3, Proposition 2.4], this functor \mathcal{F} in Eq. (3.1) defines a principal $\varpi(M, x_0)$ -bundle over M .

The above tautological principal $\varpi(M, x_0)$ -bundle over M will be denoted by $E_{\varpi(M, x_0)}$.

We also note that the restriction of the principal $\varpi(M, x_0)$ -bundle $E_{\varpi(M, x_0)}$ to the base point x_0 is canonically trivialized. This trivialization is obtained from the facts that the fiber functor for $\text{Rep}(\varpi(M, x_0))$ takes any W to the fiber of $\mathcal{F}(W)$ over x_0 (see Eq. (3.1) for \mathcal{F}), and $E_{\varpi(M, x_0)}$ is defined by \mathcal{F} . Let

$$(3.2) \quad e_0 \in (E_{\varpi(M, x_0)})_{x_0}$$

be the point that corresponds to the identity element in $\varpi(M, x_0)$ by the canonical trivialization of the $\varpi(M, x_0)$ -torsor $(E_{\varpi(M, x_0)})_{x_0}$.

As in Section 1, for any homomorphism

$$(3.3) \quad \rho : \varpi(M, x_0) \longrightarrow G$$

of group-schemes, the principal G -bundle over M obtained by extending the structure group of the principal $\varpi(M, x_0)$ -bundle $E_{\varpi(M, x_0)}$ using ρ will be denoted by E_ρ .

Theorem 3.1. *Let E_G be a principal G -bundle over M . The following three statements are equivalent.*

- (1) *There are integers $b > a \geq 1$, such that the principal G -bundle $(F_M^b)^*E_G$ is isomorphic to $(F_M^a)^*E_G$.*
- (2) *There is a homomorphism ρ as in Eq. (3.3) such that E_G is isomorphic to the principal G -bundle E_ρ .*
- (3) *The principal G -bundle E_G is strongly semistable and the following two conditions hold:*
 - *for each character χ of G , the line bundle $E_G(\chi) = E_G \times^G k \longrightarrow M$ associated to E_G for χ has the property that*

$$[c_1(E_G(\chi))c_1(\xi)^{d-1}] \in \mathbb{Z}$$

vanishes, and

- *$[c_2(\text{ad}(E_G))c_1(\xi)^{d-2}] \in \mathbb{Z}$ vanishes, where $\text{ad}(E_G)$ is the adjoint bundle of E_G .*

Proof. We will show that the first statement implies the third statement. Assume that $(F_M^b)^*E_G$ is isomorphic to $(F_M^a)^*E_G$, where $b > a \geq 1$. Then all the numerical invariants of E_G vanish. To prove that E_G is strongly semistable, first note that the vector bundle $(F_M^b)^*\text{ad}(E_G)$ is isomorphic to $(F_M^a)^*\text{ad}(E_G)$. For a coherent subsheaf $E \subset \text{ad}(E_G)$ with $\text{degree}(E) > 0$ (the degree is defined using ξ), we have

$$\text{degree}((F_M^{jb-jb+1})^*E) = p(jb - ja + 1) \cdot \text{degree}(E)$$

for the subsheaf

$$(F_M^{jb-jb+1})^*E \subset (F_M^{jb-jb+1})^*(F_M^a)^*\text{ad}(E_G) = (F_M^a)^*\text{ad}(E_G),$$

where p is the characteristic of the field k and j is any positive integer. But any given vector bundle can not contain subsheaves of arbitrarily large degrees. In particular, $(F_M^a)^*\text{ad}(E_G)$ does not contain subsheaves of arbitrarily large degrees. Therefore, we conclude that $\text{ad}(E_G)$ does not contain any subsheaf of positive degree. Hence $\text{ad}(E_G)$ is semistable. This immediately implies that the principal G -bundle E_G is semistable. Now replacing E_G by $(F_M^n)^*E_G$ in the above argument it follows that E_G is strongly semistable.

Therefore, the first statement in the theorem implies the third statement. We will now show that the third statement implies the first statement.

The family of principal G -bundles over M satisfying all the conditions in the third statement in the theorem is bounded [10, p. 533, Theorem 7.3] (see also [4]). Note that if a principal G -bundle E'_G satisfies all the conditions in the third statement, then $F_M^*E'_G$ also satisfies all these conditions. Therefore, from the finiteness of the field k we conclude that if the third statement holds, then the set of principal G -bundles $\{(F_M^i)^*E_G\}_{i \geq 1}$ contains only finitely many isomorphism classes. Hence the first statement in the theorem follows from the third statement.

Now assume that the second statement in the theorem holds. Therefore, for each G -module V , the vector bundle $E_G(V) \rightarrow M$ associated to E_G for V is essentially finite. We note that for an essentially finite vector bundle $W \rightarrow M$, there is a finite group-scheme Γ over k and a principal Γ -bundle

$$\gamma : E_\Gamma \rightarrow M$$

such that the vector bundle $\gamma^*W \rightarrow E_\Gamma$ is trivializable; this follows from [13, p. 83, Proposition 3.10]. Using this we conclude the following:

- W is strongly semistable, and
- $[c_i(W)c_1(\xi)^{d-i}] = 0$ for all $i \geq 1$.

In particular, $E_G(V)$ is strongly semistable and

$$[c_i(E_G(V))c_1(\xi)^{d-i}] = 0$$

for all $i \geq 1$. Consequently, the third statement in the theorem holds.

The assertion that the third statement implies the second statement is essentially contained in Lemma 2.1. To explain this, assume that the third statement holds.

Since the field k is finite, a theorem of Lang says that the fiber $(E_G)_{x_0}$ is a trivial G -torsor (see [7, p. 557, Theorem 2]). Fix a k -rational point

$$(3.4) \quad z_0 \in (E_G)_{x_0}.$$

Let $\text{Rep}(G)$ denote the category of all finite dimensional representations of G over k . For any $V \in \text{Rep}(G)$, let $E_G(V)$ be the vector bundle over M associated to E_G for V . The point z_0 in Eq. (3.4) defines an isomorphism

$$(3.5) \quad f_{V,z_0} : V \rightarrow E_G(V)_{x_0}$$

of vector spaces over k . More precisely, f_{V,z_0} sends any vector $w \in V$ to the image of (z_0, w) in $E_G(V)_{x_0}$ (recall that $E_G(V) := E_G \times^G V$ is a quotient of $E_G \times V$).

Let

$$(3.6) \quad \mathcal{F}_0 : \text{Rep}(G) \longrightarrow \text{Rep}(\varpi(M, x_0))$$

(see Eq. (3.1)) be the functor that sends any G -module V to the essentially finite vector bundle $E_G(V)$ associated to E_G for V (it was shown in Lemma 2.1 that $E_G(V)$ is essentially finite); recall that an essentially finite vector bundle over M gives an object of $\text{Rep}(\varpi(M, x_0))$.

Let $k\text{-mod}$ denote the category of finite dimensional vector spaces over k . Let

$$\mathcal{G} : \text{Rep}(G) \longrightarrow k\text{-mod}$$

be the fiber functor for G that sends any G -module to the underlying k -vector space. Similarly, let

$$\mathcal{H} : \text{Rep}(\varpi(M, x_0)) \longrightarrow k\text{-mod}$$

be the fiber functor for $\varpi(M, x_0)$ that sends an essentially finite vector bundle W to its fiber W_{x_0} over x_0 . For any $V \in \text{Rep}(G)$, the homomorphism f_{V,z_0} in Eq. (3.5) is an isomorphism of $\mathcal{G}(V)$ with $\mathcal{H}(\mathcal{F}_0(V))$, where \mathcal{F}_0 is constructed in Eq. (3.6).

Now we note that \mathcal{F}_0 and the isomorphisms $\{f_{V,z_0}\}_{V \in \text{Rep}(G)}$ together define a homomorphism of group-schemes

$$(3.7) \quad \rho : \varpi(M, x_0) \longrightarrow G$$

(recall that $\varpi(M, x_0)$ is defined to be the group-scheme corresponding to the neutral Tannakian category defined by the category of essentially finite vector bundles on M equipped with the fiber functor that sends any essentially finite vector bundle to its fiber over x_0).

Let E_ρ denote the principal G -bundle over M obtained by extending the structure group of the tautological principal $\varpi(M, x_0)$ -bundle $E_{\varpi(M, x_0)} \longrightarrow M$ using the homomorphism ρ in Eq. (3.7). Note that the morphism

$$\text{Id}_{E_{\varpi(M, x_0)}} \times \rho : E_{\varpi(M, x_0)} \times \varpi(M, x_0) \longrightarrow E_{\varpi(M, x_0)} \times G$$

descends to a morphism

$$(3.8) \quad \tilde{\rho} : E_{\varpi(M, x_0)} = E_{\varpi(M, x_0)} \times^{\varpi(M, x_0)} \varpi(M, x_0) \longrightarrow E_{\varpi(M, x_0)} \times^{\varpi(M, x_0)} G =: E_\rho$$

between the quotient spaces.

From the construction of the homomorphism ρ in Eq. (3.7) it follows that for any $V \in \text{Rep}(G)$, the vector bundle $E_\rho \times^G V \longrightarrow M$ associated to E_ρ for V is identified with the vector bundle $E_G(V)$ associated to E_G for V . Therefore, we get an isomorphism of the principal G -bundle E_ρ with E_G . This completes the proof of the theorem. \square

Note that the above identification of E_G with E_ρ takes the point in E_ρ defined by $(e_0, e) \in E_{\varpi(M, x_0)} \times G$, where $e \in G$ is the identity element and e_0 is the element in Eq. (3.2), to the point z_0 of E_G in Eq. (3.4).

The homomorphism ρ in Eq. (3.7) depends on the choice of z_0 . Take any $g \in G$. Let

$$\rho' : \varpi(M, x_0) \longrightarrow G$$

denote the homomorphism obtained in place of ρ after we replace z_0 by $z'_0 = z_0 g$. Then,

$$(3.9) \quad \rho' = I_{g_0} \circ \rho,$$

where I_{g_0} is the inner automorphism of G defined by $g \longmapsto g_0^{-1} g g_0$.

Consider the adjoint action of G on itself. Let

$$\text{Ad}(E_G) = E_G \times^G G \longrightarrow M$$

be the associated fiber bundle. Since the adjoint action of G on itself preserve the group structure, it follows that $\text{Ad}(E_G)$ is a group-scheme over M . Let

$$\text{Ad}(E_{\varpi(M, x_0)}) := E_{\varpi(M, x_0)} \times^{\varpi(M, x_0)} \varpi(M, x_0) \longrightarrow M$$

be the adjoint group-scheme for $E_{\varpi(M, x_0)}$. Since $E_G = E_\rho$ is an extension of structure group of $E_{\varpi(M, x_0)}$ using ρ , we have a homomorphism

$$(3.10) \quad \varphi : \text{Ad}(E_{\varpi(M, x_0)}) \longrightarrow \text{Ad}(E_G)$$

of group-schemes over M . Indeed, the morphism

$$\tilde{\rho} \times \rho : E_{\varpi(M, x_0)} \times \varpi(M, x_0) \longrightarrow E_\rho \times G,$$

where $\tilde{\rho}$ is constructed in Eq. (3.8), descends to the morphism φ in Eq. (3.10) between the quotient spaces. Now, using Eq. (3.9) it is straight forward to check that the morphism φ does not depend on the choice of the point z_0 . The image of φ is also independent of the choice of the base point x_0 .

Lemma 2.1 and Theorem 3.1 have the following corollary:

Corollary 3.2. *A vector bundle E over M is essentially finite if and only if there are integers $b > a \geq 1$, such that the vector bundle $(F_M^b)^* E$ is isomorphic to $(F_M^a)^* E$.*

Proof. Assume that $(F_M^b)^* E$ is isomorphic to $(F_M^a)^* E$ for some a and b with $b > a \geq 1$. Hence Eq. (2.5) holds. Now exactly as in the proof of Lemma 2.1, using the theorem of Lange–Stuhler and Deligne together with Proposition 2.3 of [2] we conclude that E is essentially finite.

Now assume that E is an essentially finite vector bundle of rank r . Set $G = \text{GL}(r, k)$, and substitute for E_G in Theorem 3.1 the principal $\text{GL}(r, k)$ -bundle $E_{\text{GL}(r, k)} \longrightarrow M$ given by E . Since E is essentially finite, we know that there is homomorphism

$$\rho : \varpi(M, x_0) \longrightarrow \text{GL}(r, k)$$

such that $E_{\text{GL}(r, k)}$ is isomorphic to the principal $\text{GL}(r, k)$ -bundle obtained by extending the structure group of the tautological principal $\varpi(M, x_0)$ -bundle $E_{\varpi(M, x_0)} \longrightarrow M$ using ρ ; the homomorphism ρ is given by the functor

$$\text{Rep}(\text{GL}(r, k)) \longrightarrow \text{Rep}(\varpi(M, x_0))$$

that sends a $\mathrm{GL}(r, k)$ -module V to the associated vector bundle $E_{\mathrm{GL}(r, k)}(V)$. Hence from Theorem 3.1 we know that there are integers $b > a \geq 1$, such that the principal $\mathrm{GL}(r, k)$ -bundle $(F_M^b)^* E_{\mathrm{GL}(r, k)}$ is isomorphic to $(F_M^a)^* E_{\mathrm{GL}(r, k)}$. This completes the proof of the corollary. \square

REFERENCES

- [1] Biswas, I., Parameswaran, A. J. and Subramanian, S.: Monodromy group for a strongly semistable principal bundle over a curve, *Duke Math. Jour.* **132** (2006), 1–48.
- [2] Biswas, I. and Holla, Y. I.: Comparison of fundamental group schemes of a projective variety and an ample hypersurface, *Jour. Alg. Geom.* **16** (2007), 547–597.
- [3] Bogomolov, F. A.: Holomorphic tensors and vector bundles on projective varieties, *Math. USSR-Izv.* **13** (1978), 495–555.
- [4] Coiai, F. and Holla, Y. I.: Extensions of structure groups of principal bundles in positive characteristics, *Jour. Reine Angew. Math.* **595** (2006), 1–24.
- [5] Deligne, P. and Milne, J. S.: Tannakian Categories, in: *Hodge cycles, motives, and Shimura varieties* (by P. Deligne, J. S. Milne, A. Ogus and K.-Y. Shih), pp. 101–228, Lecture Notes in Mathematics, **900**, Springer-Verlag, Berlin-Heidelberg-New York, 1982.
- [6] Giraud, J.: Cohomologie non abélienne, Die Grundlehren der mathematischen Wissenschaften, Band 179, Springer-Verlag, Berlin-New York, 1971.
- [7] Lang, S.: Algebraic groups over finite fields, *Amer. Jour. Math.* **78** (1956), 555–563.
- [8] Lange, H. and Stuhler, U.: Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe, *Math. Zeit.* **156** (1977), 73–83.
- [9] Langer, A.: Semistable sheaves in positive characteristic, *Ann. of Math.* **159** (2004), 251–276.
- [10] Langer, A.: Semistable principal G -bundles in positive characteristics, *Duke Math. Jour.* **128** (2005), 511–540.
- [11] Laszlo, Y.: A non-trivial family of bundles fixed by the square of Frobenius, *Comp. Ran. Acad. Sci. Paris* **333** (2001), 651–656.
- [12] Nori, M. V.: On the representations of the fundamental group scheme, *Compos. Math.* **33** (1976), 29–41.
- [13] Nori, M. V.: The fundamental group-scheme, *Proc. Ind. Acad. Sci. (Math. Sci.)* **91** (1982), 73–122.
- [14] Ramanan, S. and Ramanathan, A.: Some remarks on the instability flag, *Tôhoku Math. Jour.* **36** (1984), 269–291.
- [15] Ramanathan, A.: Stable principal bundles on a compact Riemann surface, *Math. Ann.* **213** (1975), 129–152.
- [16] Subramanian, S.: Strongly semistable bundles on a curve over a finite field, *Arch. Math.* **89** (2007), 68–72.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA

E-mail address: indranil@math.tifr.res.in